

Exponential Polynomials and its Applications to the Related Polynomials and Numbers

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Abstract

In this paper we use computational method based on operational point of view to prove a new generating function for exponential polynomials. We give some applications involving geometric polynomials, Apostol-Bernoulli and Apostol-Euler numbers of higher order. We obtain generalized recurrence relations and explicit expressions for these polynomials and numbers. We also study some special cases.

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1 Introduction, Definitions and Notations

The exponential polynomials, defined by

$$\phi_n(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k, \quad (1)$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is Stirling numbers of the second kind, were first studied by Grunert [23], with the operational formula

$$(xD)^n e^x = \phi_n(x) e^x, \quad (xD) = x \frac{d}{dx}.$$

These polynomials also appear in Ramanujan's unpublished notebooks. He obtained the exponential generating function of $\phi_n(x)$ as

$$\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = e^{x(e^t-1)} \quad (2)$$

and proved the relation

$$\phi_{n+1}(x) = x \left(\phi_n(x) + \phi'_n(x) \right). \quad (3)$$

We refer to [4, 7, 8, 9, 18, 24, 38, 39] for details of these polynomials. In particular $\phi_n(1)$ are known as exponential numbers (or Bell numbers) b_n , defined by ([1, 5, 15, 16, 36])

$$b_n = \phi_n(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}. \quad (4)$$

Geometric polynomials are defined by

$$w_n(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k! x^k. \quad (5)$$

These polynomials are related to the geometric series as follows

$$(xD)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{1}{1-x} w_n\left(\frac{x}{1-x}\right), \quad |x| < 1.$$

We refer to [8, 10, 11, 12, 18, 19] for details of these polynomials. By setting $x = 1$ in (5) one can obtain geometric numbers (or ordered Bell numbers, or Fubini numbers) w_n as

$$F_n = w_n(1) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k!, \quad (6)$$

([17, 18, 21]).

Geometric and exponential polynomials are connected by the relation ([8])

$$w_n(x) = \int_0^{\infty} \phi_n(x\lambda) e^{-\lambda} d\lambda. \quad (7)$$

Thus one can derive the properties of w_n from those of ϕ_n . For example, (7) can be used to obtain the exponential generating function for the geometric polynomials

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} w_n(x) \frac{t^n}{n!}. \quad (8)$$

Furthermore as a natural generalization of geometric polynomials, Boyadzhiev [8] obtain a general class of geometric polynomials as

$$w_{n,\alpha}(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} k! x^k, \quad (9)$$

using the operator

$$(xD)^n \left\{ \frac{1}{(1-x)^\alpha} \right\} = \frac{1}{(1-x)^\alpha} w_{n,\alpha} \left(\frac{x}{1-x} \right), \quad \operatorname{Re}(\alpha) > 0, \quad |x| < 1.$$

Therefore, the relation

$$w_n(x) = w_{n,1}(x), \quad n \in \mathbb{N} \cup \{0\},$$

follows directly.

The generalized Apostol-Euler polynomials of higher order are defined by means of the following generating function [28]

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < |\log(-\lambda)|), \quad (10)$$

Some particular cases of $\mathcal{E}_n^{(\alpha)}(x; \lambda)$ are

$$\begin{aligned} E_n^{(\alpha)}(x) &= \mathcal{E}_n^{(\alpha)}(x; 1) \text{ and } E_n^{(\alpha)} = E_n^{(\alpha)}(0) = \mathcal{E}_n^{(\alpha)}(0; 1), \\ \mathcal{E}_n(x; \lambda) &= \mathcal{E}_n^{(1)}(x; \lambda) \text{ and } \mathcal{E}_n(\lambda) = \mathcal{E}_n^{(1)}(0; \lambda), \\ E_n(x) &= \mathcal{E}_n^{(1)}(x; 1) \text{ and } E_n = E_n(0) = \mathcal{E}_n^{(1)}(0; 1), \end{aligned}$$

where $E_n^{(\alpha)}(x)$, $E_n^{(\alpha)}$, $\mathcal{E}_n(x; \lambda)$, $\mathcal{E}_n(\lambda)$, $E_n(x)$ and E_n denote Euler polynomials of higher order, Euler numbers of higher order, Apostol-Euler polynomials, Apostol-Euler numbers, classical Euler polynomials and classical Euler numbers, respectively. Comparing (8) and (10) one can obtain

$$w_n\left(\frac{-1}{2}\right) = E_n. \quad (11)$$

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x, \lambda)$ of higher order are defined by means of generating function [32]

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (12)$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\log \lambda|$ when $\lambda \neq 1$. As special cases we write

$$\begin{aligned} B_n^{(\alpha)}(x) &= \mathcal{B}_n^{(\alpha)}(x; 1) \text{ and } B_n^{(\alpha)} = B_n^{(\alpha)}(0) \\ \mathcal{B}_n(x; \lambda) &= \mathcal{B}_n^{(1)}(x; \lambda) \text{ and } \mathcal{B}_n(\lambda) = \mathcal{B}_n^{(1)}(0; \lambda), \\ B_n(x) &= \mathcal{B}_n^{(1)}(x; 1) \text{ and } B_n = B_n(0) = \mathcal{B}_n^{(1)}(0; 1), \end{aligned}$$

where $B_n^{(\alpha)}(x)$, $B_n^{(\alpha)}$, $\mathcal{B}_n(x; \lambda)$, $\mathcal{B}_n(\lambda)$, $B_n(x)$ and B_n denote Bernoulli polynomials of higher order, Bernoulli numbers of higher order, Apostol-Bernoulli polynomials, Apostol-Bernoulli numbers, classical Bernoulli polynomials and classical Bernoulli numbers, respectively.

Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$ have the following explicit expression [2, Eq. (3.7)]

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda - 1} \sum_{k=0}^{n-1} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} k! \left(\frac{\lambda}{1-\lambda} \right)^k, \quad \lambda \in \mathbb{C} \setminus \{1\}. \quad (13)$$

Thus comparing the above equation with (5), Apostol-Bernoulli numbers can be expressed by geometric polynomials as ([10])

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda - 1} w_{n-1} \left(\frac{\lambda}{1 - \lambda} \right), \quad \lambda \in \mathbb{C} \setminus \{1\}. \quad (14)$$

We can observe from (11) and (14) that to study the properties of geometric polynomials is vital for gaining the known and unknown properties of Bernoulli and Euler numbers which have important positions in number theory.

The starting point of this study is the formula (3), which can be written as

$$\phi_{n+1}(x) = \widehat{M} \phi_n(x),$$

where $\widehat{M} = (x + xD)$. The operator \widehat{M} can be considered as a rising operator acting on the polynomials $\phi_n(x)$. Thus $\phi_n(x)$ can be explicitly constructed the action of \widehat{M}^n on $\phi_0(x) = 1$ as

$$\phi_n(x) = \widehat{M}^n \{1\}.$$

From this motivation we introduce an operational formula for an appropriate function f as

$$\widehat{M}^n f(x) = \sum_{k=0}^n \binom{n}{k} \phi_{n-k}(x) (xD)^k f(x),$$

in order to give a direct proof of generalized recurrence relation

$$\phi_{n+m}(x) = \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j^{n-k} x^j \phi_k(x), \quad (15)$$

and to obtain a new generating function for exponential polynomials. Because of the close relationship with exponential polynomials, we consider the general geometric polynomials $w_{n,\alpha}(x)$, and obtain a generating function as

$$\sum_{n=0}^{\infty} w_{n+m,\alpha}(x) \frac{t^n}{n!} = \left(\frac{1}{1 - x(e^t - 1)} \right)^{\alpha} w_{m,\alpha} \left(\frac{x e^t}{1 - x(e^t - 1)} \right).$$

With the help of this generating function we arrive two conclusions. First we obtain relations between $w_{n,\alpha}(x)$ and Apostol-Euler, Apostol-Bernoulli numbers of higher order which generalize the equations (11) and (14). Then we get a generalized recurrence relation for general geometric polynomials

$$w_{n+m}^{(\alpha)}(x) = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{\alpha + k - 1}{k} k! k^{n-j} x^k w_j^{(\alpha+k)}(x).$$

Dealing with recurrence relations of Apostol-Euler and Apostol-Bernoulli numbers of higher order, which are in particular useful in computing special values of these numbers is considerably important and studied by many mathematicians

(see [13, 14, 26, 28, 29, 32, 33, 35, 37]). Therefore, the connections between $w_{n,\alpha}(x)$ and these numbers enable us to obtain generalized recurrence relations for Apostol-Euler and Apostol-Bernoulli numbers of higher order. We also give new generating functions of these numbers. Furthermore we deduce their special cases involving Euler and Bernoulli numbers of higher order, Apostol-Euler and Apostol-Bernoulli numbers and classical Euler and Bernoulli numbers.

The organization of this paper is as follows. In section 2 we examine $\phi_{n+m}(x)$ for $n, m \in \mathbb{N} \cup \{0\}$. We give a different proof for (15) and obtain a new generating function. In section 3 we deal with general geometric polynomials, Apostol-Euler and Apostol-Bernoulli numbers of higher order and find new generating functions and recurrence relations.

Now we state our results.

2 An Operational Formula and Its Results for Exponential Polynomials

In this section we obtain an operational formula involving exponential polynomials and a new generating function of these polynomials. First we give the following theorem.

Theorem 1 *The following operational formula holds for an appropriate function f :*

$$\widehat{M}^n f(x) = \sum_{k=0}^n \binom{n}{k} \phi_{n-k}(x) (xD)^k f(x), \quad (16)$$

where $\widehat{M} = (x + xD)$.

Proof. Proof follows from induction on n . Assume the statement holds for n ; that is,

$$\widehat{M}^n f(x) = \sum_{k=0}^n \binom{n}{k} \phi_{n-k}(x) (xD)^k f(x).$$

We need to show that this is true for $n+1$.

$$\begin{aligned} \widehat{M}^{n+1} f(x) &= \widehat{M} \widehat{M}^n f(x) \\ &= \sum_{k=0}^n \binom{n}{k} x \phi_{n-k}(x) (xD)^k f(x) + \sum_{k=0}^n \binom{n}{k} (xD) \left[\phi_{n-k}(x) (xD)^k f(x) \right]. \end{aligned}$$

Using (3) we have

$$\begin{aligned} \widehat{M}^{n+1} f(x) &= \sum_{k=0}^n \binom{n}{k} \phi_{n+1-k}(x) (xD)^k f(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} \phi_{n+1-k}(x) (xD)^k f(x) \\ &= \phi_{n+1}(x) + (xD)^{n+1} f(x) + \sum_{k=1}^n \left[\binom{n}{k} + \binom{n}{k-1} \right] \phi_{n+1-k}(x) (xD)^k f(x). \end{aligned}$$

Finally using the well-known binomial recursion relation

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

gives the statement is true for $n+1$. ■

Setting $f(x) = \phi_m(x)$ in (16) and using the definition of exponential polynomials given by (1) we get the following identity which also occurs in [6, 9, 22].

Corollary 2 For $n, m \in \mathbb{N} \cup \{0\}$,

$$\phi_{n+m}(x) = \sum_{k=0}^n \sum_{j=0}^m \binom{n}{k} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j^{n-k} x^j \phi_k(x). \quad (17)$$

We note that setting $x = 1$, (17) coincides with the formula which was given by Spivey ([34]) for the Bell numbers.

A natural question is to find the generating function for $\phi_{n+m}(x)$. To answer this we need the following theorem.

Theorem 3 Let

$$g(x) = \sum_{k=0}^{\infty} c_k x^k$$

be the formal series expansion of $g(x)$. Then we have

$$e^{t\widehat{M}}g(x) = e^{x(e^t-1)}g(xe^t). \quad (18)$$

Proof. Acting the operator to $g(x)$ we get

$$e^{t\widehat{M}}g(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \widehat{M}^n c_k x^k. \quad (19)$$

For $f(x) = x^k$, (16) and (19) show that

$$\begin{aligned} e^{t\widehat{M}}g(x) &= \left[\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} \right] \left[\sum_{k=0}^{\infty} c_k \left(\sum_{m=0}^{\infty} \frac{k^m t^m}{m!} \right) x^k \right] \\ &= e^{x(e^t-1)}g(xe^t). \end{aligned}$$

■

For a use of this operational formula, we set $g(x) = \phi_m(x)$ in (18). Then we obtain the following theorem which gives a new generating function for the exponential polynomials.

Corollary 4 For $m \in \mathbb{N} \cup \{0\}$, the exponential polynomials have the following generating function

$$\sum_{n=0}^{\infty} \phi_{n+m}(x) \frac{t^n}{n!} = e^{x(e^t-1)} \phi_m(xe^t), \quad t \in \mathbb{C}. \quad (20)$$

Since $\phi_0(x) = 1$, (20) reduce to (2), obtained by Ramanujan. Moreover setting $x = 1$ and $t = ki\pi$ ($k \in \mathbb{Z}$) in (20) we get the value of infinite summation of Bell numbers

$$\sum_{n=0}^{\infty} b_{n+m} \frac{(ki\pi)^n}{n!} = \begin{cases} b_m & , \text{ if } k \text{ even} \\ e^{-2} \tilde{b}_m & , \text{ if } k \text{ odd} \end{cases}$$

where \tilde{b}_m is the complementary Bell numbers (or Uppuluri-Carpenter numbers) ([3, 7, 40]).

3 Applications to the Related Polynomials and Numbers

In this section we give some applications of (20) for several polynomials and numbers, which are closely related to $\phi_n(x)$.

Formula (7) can be extended to the integral representation

$$w_{n,\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{\alpha-1} \phi_n(x\lambda) e^{-\lambda} d\lambda, \quad (21)$$

which is verified immediately by using (1), (9) and Euler's integral representation for the gamma function. Thus the exponential generating function for general geometric polynomials can be found by writing (20) in the form

$$\sum_{n=0}^{\infty} \phi_{n+m}(x\lambda) \frac{t^n}{n!} = e^{x\lambda(e^t-1)} \phi_m(x\lambda e^t),$$

then multiplying both sides by $\lambda^{\alpha-1} e^{-\lambda}$ and integrating for λ from zero to infinity. In the view of (21) this gives

$$\sum_{n=0}^{\infty} w_{n+m,\alpha}(x) \frac{t^n}{n!} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{\alpha-1} e^{-\lambda(1-x(e^t-1))} \phi_m(x\lambda e^t) d\lambda,$$

and this equation leads the following result by a simple change of variable.

Theorem 5 *We have the following generating function:*

$$\sum_{n=0}^{\infty} w_{n+m,\alpha}(x) \frac{t^n}{n!} = \left(\frac{1}{1-x(e^t-1)} \right)^{\alpha} w_{m,\alpha} \left(\frac{x e^t}{1-x(e^t-1)} \right),$$

where $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$.

For $\alpha = 1$ in the above theorem we have the following generating function for $w_n(x)$.

Corollary 6 *We have*

$$\sum_{n=0}^{\infty} w_{n+m}(x) \frac{t^n}{n!} = \frac{1}{1-x(e^t-1)} w_m \left(\frac{xe^t}{1-x(e^t-1)} \right). \quad (22)$$

Since $w_0(x) = 1$ (22) reduce to (8) obtained by Boyadzhiev. Also, for $m = 0$, we have $w_{0,\alpha}(x) = 1$, which leads the following generating function for the general geometric polynomials as

$$\sum_{n=0}^{\infty} w_{n,\alpha}(x) \frac{t^n}{n!} = \left(\frac{1}{1-x(e^t-1)} \right)^\alpha, \quad (23)$$

which is also given in ([20]) by different way.

From now on we use the notation $w_n^{(\alpha)}(x)$ instead of $w_{n,\alpha}(x)$ since (23) has some similarities with the generating functions of the generalized Apostol-Bernoulli and Apostol-Euler numbers of higher order, for example

$$w_{n,\alpha} \left(\frac{-\lambda}{\lambda+1} \right) = \left(\frac{\lambda+1}{2} \right)^\alpha \mathcal{E}_n^{(\alpha)}(\lambda), \quad \lambda \in \mathbb{C} \quad (24)$$

and

$$w_{n,l} \left(\frac{-\lambda}{\lambda-1} \right) = \frac{(\lambda-1)^l}{l!} \binom{n+l}{l}^{-1} \mathcal{B}_{n+l}^{(l)}(\lambda), \quad \lambda \in \mathbb{C} \setminus \{1\} \text{ and } \alpha = l \in \mathbb{N}. \quad (25)$$

Therefore one can derive properties of generalized Apostol-Bernoulli and Apostol-Euler numbers of higher order from general geometric polynomials. Moreover setting $\alpha = \lambda = 1$ in (24) and $l = 1$ in (25) we have (11) and (14) respectively.

Now we state some applications of Theorem 5. First we have the following corollary.

Corollary 7 *We have the generating functions for Apostol-Euler and Apostol-Bernoulli numbers of higher order as:*

$$\sum_{n=0}^{\infty} \mathcal{E}_{n+m}^{(\alpha)}(\lambda) \frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1} \right)^\alpha w_m^{(\alpha)} \left(\frac{-\lambda e^t}{\lambda e^t + 1} \right), \quad (26)$$

$$\sum_{n=0}^{\infty} \binom{n+m+l}{l}^{-1} \mathcal{B}_{n+m+l}^{(l)}(\lambda) \frac{t^n}{n!} = l! \left(\frac{1}{\lambda e^t - 1} \right)^l w_m^{(l)} \left(\frac{-\lambda e^t}{\lambda e^t - 1} \right). \quad (27)$$

Setting $\lambda = 1$, $\alpha = 1$ and $\lambda = \alpha = 1$ in (26) we have the following generating functions for Euler numbers of higher order, Apostol-Euler numbers and classical Euler numbers

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n+m}^{(\alpha)} \frac{t^n}{n!} &= \left(\frac{2}{e^t + 1} \right)^{\alpha} w_m^{(\alpha)} \left(\frac{-e^t}{e^t + 1} \right), \\ \sum_{n=0}^{\infty} \mathcal{E}_{n+m}(\lambda) \frac{t^n}{n!} &= \frac{2}{\lambda e^t + 1} w_m \left(\frac{-\lambda e^t}{\lambda e^t + 1} \right), \\ \sum_{n=0}^{\infty} E_{n+m} \frac{t^n}{n!} &= \frac{2}{e^t + 1} w_m \left(\frac{-e^t}{e^t + 1} \right),\end{aligned}$$

respectively. Similarly, setting appropriate parameters in (27) we have the following results for Bernoulli numbers of higher order, Apostol-Bernoulli numbers and classical Bernoulli numbers

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{n+m+l}{l}^{-1} B_{n+m+l}^{(l)} \frac{t^n}{n!} &= l! \left(\frac{1}{e^t - 1} \right)^l w_m^{(l)} \left(\frac{-e^t}{e^t - 1} \right) \\ \sum_{n=0}^{\infty} \frac{\mathcal{B}_{n+m+1}(\lambda)}{n+m+1} \frac{t^n}{n!} &= \frac{1}{\lambda e^t - 1} w_m \left(\frac{-\lambda e^t}{\lambda e^t - 1} \right) \\ \sum_{n=0}^{\infty} \frac{B_{n+m+1}}{n+m+1} \frac{t^n}{n!} &= \frac{1}{e^t - 1} w_m \left(\frac{e^t}{1 - e^t} \right),\end{aligned}$$

respectively. Finally, for $t = i\pi$ in (26) and (27) we have the values of infinite summation for Apostol-Euler and Apostol-Bernoulli numbers of higher order

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_{n+m}^{(l)}(\lambda) \frac{(i\pi)^n}{n!} &= \frac{(-2)^l}{l!} \binom{m+l}{l}^{-1} \mathcal{B}_{m+l}^{(l)}(\lambda), \\ \sum_{n=0}^{\infty} \binom{n+m+l}{l}^{-1} \mathcal{B}_{n+m+l}^{(l)}(\lambda) \frac{(i\pi)^n}{n!} &= \left(\frac{-1}{2} \right)^l l! \mathcal{E}_m^{(l)}(\lambda).\end{aligned}$$

Similarly for $t = 2i\pi$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_{n+m}^{(\alpha)}(\lambda) \frac{(2i\pi)^n}{n!} &= \mathcal{E}_m^{(\alpha)}(\lambda), \\ \sum_{n=1}^{\infty} \binom{n+m+l}{l}^{-1} \mathcal{B}_{n+m+l}^{(l)}(\lambda) \frac{(2i\pi)^n}{n!} &= 0.\end{aligned}$$

The second application of Theorem 5 is given in the following theorem.

Theorem 8 *We have the following generalized recurrence relation for general geometric polynomials*

$$w_{n+m}^{(\alpha)}(x) = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{\alpha+k-1}{k} k! k^{n-j} x^k w_j^{(\alpha+k)}(x), \quad (28)$$

where $n, m \in \mathbb{N} \cup \{0\}$.

Proof. Using (9) in (23), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} w_{n+m}^{(\alpha)}(x) \frac{t^n}{n!} &= \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} k! x^k \left(\frac{1}{1-x(e^t-1)} \right)^{\alpha+k} e^{kt} \\
&= \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} k! x^k \left[\sum_{j=0}^{\infty} w_j^{(\alpha+k)}(x) \frac{t^j}{j!} \right] \left[\sum_{n=0}^{\infty} \frac{k^n t^n}{n!} \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{\alpha+k-1}{k} k! k^{n-j} x^k w_j^{(\alpha+k)}(x) \right] \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives the desired equation. ■

We note that Equation (28) is also given in ([20]) however the proofs are different. Moreover for $n = 0$, (28) reduces to (9). Other results which can be drawn from Theorem 8 can be listed as follows. First we deal with geometric polynomials. Setting $\alpha = 1$ in (28) we have

$$w_{n+m}(x) = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} k! k^{n-j} x^k w_j^{(k+1)}(x).$$

Then using (9) in the above equation we get the following corollary.

Corollary 9 *The following explicit expression holds for geometric polynomials*

$$w_{n+m}(x) = \sum_{k=0}^m \sum_{j=0}^n \sum_{i=0}^j \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} k^{n-j} (i+k)! x^k, \quad (29)$$

where $n, m \in \mathbb{N} \cup \{0\}$.

Setting $x = 1$ in (29) yields an explicit expression for the geometric numbers as

$$F_{n+m} = \sum_{k=0}^m \sum_{j=0}^n \sum_{i=0}^j \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} k^{n-j} (i+k)!. \quad (30)$$

For $n = 0$ or $m = 0$, (30) reduces to (5).

Now we mention the results for generalized Apostol-Euler numbers of higher order. Setting $x = \frac{-\lambda}{\lambda+1}$ in (28) and using (24) we have the following corollary.

Corollary 10 *The following generalized recurrence relation holds for Apostol-Euler numbers of higher order*

$$\mathcal{E}_{n+m}^{(\alpha)}(\lambda) = \sum_{k=0}^m \sum_{j=0}^n \binom{n}{j} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \frac{(-\lambda)^k k! k^{n-j}}{2^k} \mathcal{E}_j^{(\alpha+k)}(\lambda). \quad (31)$$

For the natural consequences of the above corollary, setting $\lambda = 1$, $\alpha = 1$ and $\lambda = \alpha = 1$ in (31) we obtain the following generalized recurrence relations for Euler numbers of higher order, Apostol-Euler numbers and classical Euler numbers

$$E_{n+m}^{(\alpha)} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{\alpha+k-1}{k} \frac{k! k^{n-j}}{(-2)^k} E_j^{(\alpha+k)}, \quad (32)$$

$$\mathcal{E}_{n+m}(\lambda) = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \frac{(-\lambda)^k k^{n-j} k!}{2^k} \mathcal{E}_j^{(k+1)}(\lambda), \quad (33)$$

$$E_{n+m} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \frac{(-1)^k k!}{2^k} E_j^{(k+1)}, \quad (34)$$

respectively.

Remark 11 If $n = 0$ in (31) and if we use $\mathcal{E}_0^{(\alpha)}(\lambda) = 2^\alpha (\lambda + 1)^{-\alpha}$, we get the following explicit expression for Apostol-Euler numbers of higher order

$$\mathcal{E}_m^{(\alpha)}(\lambda) = 2^\alpha \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \frac{k! (-\lambda)^k}{(\lambda + 1)^{\alpha+k}}. \quad (35)$$

Furthermore, for $\lambda = 1$, $\alpha = 1$ and $\lambda = \alpha = 1$ in (35) we obtain explicit expressions for Euler numbers of higher order, Apostol-Euler numbers and classical Euler numbers as

$$E_m^{(\alpha)} = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \frac{(-1)^k k!}{2^k}, \quad (36)$$

$$\mathcal{E}_m(\lambda) = 2 \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{(-\lambda)^k k!}{(\lambda + 1)^{k+1}}, \quad (37)$$

$$E_m = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{(-1)^k k!}{2^k}, \quad (38)$$

respectively. We refer [28] for details. Besides, (38) is also given by [11] by different mean.

Finally we mention the results of Theorem 8 for generalized Apostol-Bernoulli numbers of higher order. Setting $x = \frac{-\lambda}{\lambda-1}$ in (28) and using (25) we have the following corollary.

Corollary 12 The following generalized recurrence relation holds for Apostol-Bernoulli numbers of higher order:

$$\frac{\mathcal{B}_{n+m+l}^{(l)}(\lambda)}{\binom{n+m+l}{l} l} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{l+k+j}{j}^{-1} \frac{(-\lambda)^k k^{n-j}}{l+k} \mathcal{B}_{l+k+j}^{(l+k)}(\lambda). \quad (39)$$

Setting $\lambda = 1$, $l = 1$ and $\lambda = l = 1$ in (39) we obtain generalized recurrence relations for Bernoulli numbers of higher order, Apostol-Bernoulli numbers and classical Bernoulli numbers

$$\frac{B_{n+m+l}^{(l)}}{\binom{n+m+l}{l}l} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{l+k+j}{j}^{-1} \frac{(-1)^k k^{n-j}}{l+k} B_{l+k+j}^{(l+k)}, \quad (40)$$

$$\frac{\mathcal{B}_{n+m+1}(\lambda)}{(n+m+1)} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{k+j+1}{j}^{-1} \frac{(-\lambda)^k k^{n-j}}{k+1} \mathcal{B}_{j+k+1}^{(k+1)}(\lambda), \quad (41)$$

$$\frac{B_{n+m+1}}{(n+m+1)} = \sum_{k=0}^m \sum_{j=0}^n \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n}{j} \binom{k+j+1}{j}^{-1} \frac{(-1)^k k^{n-j}}{k+1} B_{j+k+1}^{(k+1)}, \quad (42)$$

respectively. Besides, for $n = 0$ in (40) we have a recurrence relation for Bernoulli numbers of higher order involving Bernoulli numbers of higher order in the following corollary.

Corollary 13 *Bernoulli numbers of higher order hold*

$$B_{m+l}^{(l)} = l \binom{m+l}{l} \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{(-1)^k k!}{(l+k)} B_{l+k}^{(l+k)}. \quad (43)$$

Applying Stirling transform (see [31] for details) to (15) we have another recurrence relation as

$$B_{m+l}^{(m+l)} = \frac{(l+m)}{m!l} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \binom{k+l}{l}^{-1} B_{k+l}^{(l)}. \quad (44)$$

It is good to note that for $l = 1$ in (44) we get

$$B_{m+1}^{(m+1)} = \frac{m+1}{m!} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{B_{k+1}}{k+1}, \quad (45)$$

which is the special case of the formula given by Nörlund in [30].

Remark 14 *For $n = 0$ in (39) we obtain a recurrence relation for Apostol-Bernoulli numbers of higher order*

$$\mathcal{B}_{m+l}^{(l)}(\lambda) = l \binom{m+l}{l} \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{(-\lambda)^k k!}{(l+k)} \mathcal{B}_{l+k}^{(l+k)}(\lambda). \quad (46)$$

Replacing $m+l$ with n and setting $\mathcal{B}_k^{(k)}(\lambda) = \frac{k!}{(\lambda-1)^k}$ in (46) we obtain the explicit representation

$$\mathcal{B}_n^{(l)}(\lambda) = l! \binom{n}{l} \sum_{k=0}^{n-l} \left\{ \begin{matrix} n-l \\ k \end{matrix} \right\} \binom{l+k-1}{k} \frac{(-\lambda)^k k!}{(\lambda-1)^{l+k}}, \quad (47)$$

which is given by Srivastava et al in [32, Eq. (27)]. Moreover Apostol's formula (13) is a special case of the formula (41) when $n = 0$ and $l = 1$.

As we mentioned before, calculating the values of the Apostol-Euler and Apostol-Bernoulli polynomials of higher order is an active working area. In order to give effective calculation formulas for these polynomials we first need the following proposition.

Proposition 15 *We have the following generalized recurrence relations:*

$$\mathcal{E}_{n+m}^{(\alpha)}(\lambda) = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \left(\frac{-\lambda}{2} \right)^k k! \mathcal{E}_n^{(\alpha+k)}(k; \lambda), \quad (48)$$

$$\frac{\mathcal{B}_{n+m+l}^{(l)}(\lambda)}{\binom{n+m+l}{l}} = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{n+l+k}{n}^{-1} \frac{l(-\lambda)^k}{(l+k)} \mathcal{B}_{n+l+k}^{(k+l)}(k; \lambda). \quad (49)$$

Proof. From (9) we can write (26) as

$$\sum_{n=0}^{\infty} \mathcal{E}_{n+m}^{(\alpha)}(\lambda) \frac{t^n}{n!} = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \frac{k!(-\lambda)^k}{2^k} \left(\frac{2}{\lambda e^t + 1} \right)^{\alpha+k} e^{kt}.$$

Using (10) in the last part of the above equation we get

$$\sum_{n=0}^{\infty} \mathcal{E}_{n+m}^{(\alpha)}(\lambda) \frac{t^n}{n!} = \sum_{k=0}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \binom{\alpha+k-1}{k} \frac{k!(-\lambda)^k}{2^k} \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha+k)}(k; \lambda) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ we get the desired equation (48).

Equation (49) can be proved the same method by using (12). ■

Equations (35) and (46) are the special case $n = 0$ of the Proposition 15. Now it is better to improve the equations (48) and (49) as generalized recurrence relations for these polynomials by induction in the following theorem.

Theorem 16 *We have the following generalized recurrence relations for Apostol-Euler and Apostol-Bernoulli polynomials of higher order:*

$$\begin{aligned} \mathcal{E}_n^{(\alpha+m)}(m; \lambda) &= \frac{(-1)^n}{\lambda^n} \mathcal{E}_n^{(\alpha+m)}(\alpha; \lambda^{-1}) \\ &= \frac{(2\lambda^{-1})^m}{m!} \binom{\alpha+m-1}{m}^{-1} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \mathcal{E}_{n+k}^{(\alpha)}(\lambda), \end{aligned} \quad (50)$$

$$\begin{aligned} \mathcal{B}_{n+m+l}^{(m+l)}(m; \lambda) &= \frac{(-1)^{n+m+l}}{\lambda^{n+m+l}} \mathcal{B}_{n+m+l}^{(m+l)}(l; \lambda^{-1}) \\ &= \frac{(l+m)}{l(-\lambda)^m} \binom{n+m+l}{l} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \binom{n+l+k}{l}^{-1} \mathcal{B}_{n+l+k}^{(l)}(\lambda), \end{aligned} \quad (51)$$

where $n, m \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{N}$, $\lambda \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$ such that $\text{Re}(\alpha) > 0$.

Proof. For $m = 1$ in (48) we have

$$\mathcal{E}_n^{(\alpha+1)}(1; \lambda) = 2\lambda^{-1} \binom{\alpha}{1}^{-1} \mathcal{E}_{n+1}^{(\alpha)}(\lambda) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2(-\lambda)^{n-1} \binom{\alpha}{1}^{-1} \mathcal{E}_{n+1}^{(\alpha)}(\lambda). \quad (52)$$

Using (52) for $m = 2$ in (49) we get

$$\mathcal{E}_n^{(\alpha+2)}(2; \lambda) = \frac{(2\lambda^{-1})^2}{2!} \binom{\alpha}{2}^{-1} \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \mathcal{E}_{n+2}^{(\alpha)}(\lambda) - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathcal{E}_{n+1}^{(\alpha)}(\lambda) \right\}. \quad (53)$$

Apostol-Euler polynomials of higher order satisfy the recurrence relation [41, Eq.(1.16)]

$$\frac{\alpha\lambda}{2} \mathcal{E}_n^{(\alpha+1)}(x+1; \lambda) = x \mathcal{E}_n^{(\alpha)}(x; \lambda) - \mathcal{E}_{n+1}^{(\alpha)}(x; \lambda).$$

Thereby, setting $x = m$, and $\alpha + m$ for α we get

$$\mathcal{E}_n^{(\alpha+m+1)}(m+1; \lambda) = \frac{2m}{(\alpha+m)\lambda} \mathcal{E}_n^{(\alpha+m)}(m; \lambda) - \frac{2}{(\alpha+m)\lambda} \mathcal{E}_{n+1}^{(\alpha+m)}(m; \lambda). \quad (54)$$

Multiplying both sides of (50) by $\frac{2m}{(\alpha+m)\lambda}$ we get

$$\frac{2m}{(\alpha+m)\lambda} \mathcal{E}_n^{(\alpha+m)}(m; \lambda) = \frac{(2\lambda^{-1})^{m+1}}{(m+1)!} \binom{\alpha+m}{m+1}^{-1} \sum_{k=0}^m (-1)^k m \begin{bmatrix} m \\ k \end{bmatrix} \mathcal{E}_{n+k}^{(\alpha)}(\lambda). \quad (55)$$

Moreover taking $n+1$ for n in (50) and multiplying both sides by $\frac{2}{(\alpha+m)\lambda}$ we have

$$\frac{2}{(\alpha+m)\lambda} \mathcal{E}_{n+1}^{(\alpha+m)}(m; \lambda) = \frac{(2\lambda^{-1})^{m+1}}{(m+1)!} \binom{\alpha+m}{m+1}^{-1} \sum_{k=1}^{m+1} (-1)^k \begin{bmatrix} m \\ k-1 \end{bmatrix} \mathcal{E}_{n+k}^{(\alpha)}(\lambda). \quad (56)$$

Subtracting (56) from (55), using the well-known recurrence relation for Stirling number of the first kind

$$\begin{bmatrix} m+1 \\ k \end{bmatrix} = m \begin{bmatrix} m \\ k \end{bmatrix} + \begin{bmatrix} m \\ k-1 \end{bmatrix} \quad (57)$$

and the identity [28]

$$\mathcal{E}_n^{(\alpha)}(\alpha - x; \lambda) = \frac{(-1)^n}{\lambda^n} \mathcal{E}_n^{(\alpha)}(x; \lambda^{-1}),$$

we have the statement is true for $m+1$.

Equation (51) can be proved by the same method by using the following recurrence relation for Apostol-Bernoulli polynomials of higher order [32, Eq.(62)]

$$\alpha \lambda \mathcal{B}_n^{(\alpha+1)}(x+1; \lambda) = n x \mathcal{B}_{n-1}^{(\alpha)}(x; \lambda) + (\alpha - n) \mathcal{B}_n^{(\alpha)}(x; \lambda).$$

■

Now we mention some special cases of Theorem 16 for Apostol-Euler and Euler polynomials of higher order. For $\lambda = 1$ we have a recurrence relation for Euler polynomials of higher order involving Euler numbers of higher order as in the following corollary.

Corollary 17 *We have*

$$E_n^{(\alpha+m)}(m) = (-1)^n E_n^{(\alpha+m)}(\alpha) = \frac{(2)^m}{m! \binom{\alpha+m-1}{m}} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} E_{n+k}^{(\alpha)}. \quad (58)$$

Setting $\alpha = 1$ and $\alpha = \lambda = 1$ in (50) we have recurrence relations for Apostol-Euler polynomials of higher order and Euler polynomials of higher order involving Apostol-Euler and classical Euler numbers

$$\mathcal{E}_n^{(m+1)}(m; \lambda) = \frac{(-1)^n}{\lambda^n} \mathcal{E}_n^{(m+1)}(1; \lambda^{-1}) = \frac{(2\lambda^{-1})^m}{m!} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \mathcal{E}_{n+k}(\lambda), \quad (59)$$

$$E_n^{(m+1)}(m) = (-1)^n E_n^{(m+1)}(1) = \frac{(2)^m}{m!} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} E_{n+k}, \quad (60)$$

respectively. We also obtain closed form for some finite summations from Theorem 16. Setting $\mathcal{E}_0^{(\alpha)}(x; \lambda) = 2^\alpha (\lambda + 1)^{-\alpha}$ in (50) we have

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \mathcal{E}_k^{(\alpha)}(\lambda) = \frac{2^\alpha \lambda^m m!}{(\lambda + 1)^{\alpha+m}} \binom{\alpha + m - 1}{m}. \quad (61)$$

Furthermore for $\lambda = 1$, $\alpha = 1$ and $\lambda = \alpha = 1$ (61) we obtain

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} E_k^{(\alpha)} = \frac{m!}{2^m} \binom{\alpha + m - 1}{m}, \quad (62)$$

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \mathcal{E}_k(\lambda) = \frac{2\lambda^m m!}{(\lambda + 1)^{m+1}}, \quad (63)$$

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} E_k = \frac{m!}{2^m}, \quad (64)$$

respectively. It is good note that the above finite summations can also be obtained by applying Stirling transform to the identities in Remark 11.

Now, we mention some special cases of Theorem 16 for Apostol-Bernoulli and Bernoulli polynomials of higher order. For $\lambda = 1$ we have the following recurrence relation for Bernoulli polynomials of higher order involving Bernoulli numbers of higher order.

Corollary 18 *We have*

$$\begin{aligned} B_{n+m+l}^{(m+l)}(m) &= (-1)^{n+m+l} B_{n+m+l}^{(m+l)}(l) \\ &= \frac{(l+m)}{l} \binom{n+m+l}{l} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \binom{n+l+k}{l}^{-1} B_{n+l+k}^{(l)}. \end{aligned} \quad (65)$$

Setting $l = 1$ and $\lambda = l = 1$ in (51) we have following recurrence relations for Apostol-Bernoulli and Bernoulli polynomials of higher order involving Apostol-Bernoulli and classical Bernoulli numbers

$$\begin{aligned}\mathcal{B}_{n+m+1}^{(m+1)}(m; \lambda) &= \frac{(-1)^{n+m+1}}{\lambda^{n+m+1}} \mathcal{B}_{n+m+1}^{(m+1)}(1; \lambda^{-1}) \\ &= \frac{(m+1)(n+m+1)}{(-\lambda)^m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{\mathcal{B}_{n+k+1}(\lambda)}{n+k+1},\end{aligned}\quad (66)$$

$$\begin{aligned}B_{n+m+1}^{(m+1)}(m) &= (-1)^{n+m+1} B_{n+m+1}^{(m+1)}(1) \\ &= (m+1)(n+m+1) \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{B_{n+k+1}}{n+k+1},\end{aligned}\quad (67)$$

respectively. Moreover for $n = 0$ setting $\mathcal{B}_{m+l}^{(m+l)}(x; \lambda) = (m+l)! (\lambda - 1)^{-m-l}$ in (51) we have the following finite summation

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \binom{l+k}{l}^{-1} \mathcal{B}_{l+k}^{(l)}(\lambda) = \frac{l(m+l-1)! (-\lambda)^m}{(\lambda-1)^{m+l}} \binom{m+l}{l}^{-1}. \quad (68)$$

Thus for $l = 1$ in (68) we have

$$\sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{\mathcal{B}_{k+1}(\lambda)}{k+1} = \frac{m! (-\lambda)^m}{(m+1)(\lambda-1)^{m+1}}.$$

The above finite summations can also be obtained by applying Stirling transform to the identities in Remark 14. Besides, for $n = 0$ in (65) we have

$$B_{m+l}^{(m+l)}(m) = (-1)^{m+l} B_{m+l}^{(m+l)}(l) = \frac{(l+m)}{l} \binom{m+l}{l} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \binom{l+k}{l}^{-1} B_{l+k}^{(l)}. \quad (69)$$

When we compare (44) with (69) we obtain

$$B_{m+l}^{(m+l)}(m) = (-1)^{m+l} B_{m+l}^{(m+l)}(l) = \binom{m+l}{l}^{-1} m! B_{m+l}^{(m+l)}.$$

Replacing $m+l$ with n gives

$$B_n^{(n)}(n-l) = (-1)^n B_n^{(n)}(l) = \binom{n}{l}^{-1} (n-l)! B_n^{(n)}. \quad (70)$$

Furthermore for $l = 1$ we have

$$B_n^{(n)}(1) = \frac{(-1)^n (n-1)!}{n} B_n^{(n)}, \quad n \geq 1. \quad (71)$$

To the writer's knowledge, this relationship between $B_n^{(n)}$ and $B_n^{(n)}(l)$ has not been pointed out before. Howard deal with relationship between $B_n^{(n)}$ and

$B_n^{(n)}(1)$ and get the relations [25, Eq. (2.17)]

$$B_n^{(n)}(1) = \frac{1}{1-n} B_n^{(n-1)} \text{ and } B_n^{(n)}(1) = n!c_n, \quad (72)$$

where c_n is the Bernoulli numbers of the second kind defined by Jordan in [27]. Also by comparing (71) with (72) gives a connection as

$$c_n = \frac{(-1)^n}{n^2} B_n^{(n)}.$$

Finally, for $l = 1$ in (69) we get a recurrence relation for Bernoulli polynomials of higher order involving classical Bernoulli numbers as

$$B_{m+1}^{(m+1)}(m) = (-1)^{m+1} B_{m+1}^{(m+1)}(1) = m(m+1) \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \frac{B_{k+1}}{k+1}.$$

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